Singular Perturbation Model Governing Potential Flow of a Real Fluid Over Thin Airfoils

Omar EL-AAJINE[1,2],Nasser EDDEGDAG[2], Aze-eddine NAAMANE[1,2], Mohammed RADOUANI[2], Benaissa EL FAHIME[2]

[1]*Royal Air Force Academy, Marrakesh, 40000, Morocco.*

[2]*ENSAM, Moulay Ismail University, Meknes, 15290, Morocco.*

o.elaajine@edu.umi.ac.ma

Abstract— The potential-flow theory is a well-established theory. However, this theory fails to predict drag due to the assumption of inviscid flow known as the "D'Alembert's paradox". Existing models treat special cases of inviscid or incompressible flows.

Therefore, we propose an alternative potential-flow model derived from the conservation laws of fluid mechanics. We consider a high speed laminar, isentropic and irrotational steady flow of a real, Newtonian, and non-heating fluid over thin airfoils in a 2D curvilinear coordinate system. The developed model, using asymptotic & perturbation methods, is a singular perturbation equation that takes into consideration both compressibility and viscous effects.

Keywords— Compressible viscous fluid, potential-flow theory, laminar boundary layer, High Reynolds number, asymptotic & perturbation methods.

I. Introduction

The D'alembert paradox proves that for incompressible and inviscid Potential flow the movement through a fluid is without resistance, which is against experiments of real flow [1-2].The official resolution of the paradox blames the assumption of inviscid flow.

A first simplification is introduced by Euler equations, governing incompressible inviscid fluid flows. The basic idea was to solve the Euler equations and thereby predict fluid flow.

Saint-Venant suggested that instead of the Euler equations, one should consider the Navier-Stokes equations including forces from viscosity. Especially, the slip boundary condition of the Euler equations allowing fluid particles to slide along the boundary of the moving body without friction, should be replaced by a no-slip condition forcing fluid particles on the boundary to slow down to zero velocity [3]

This theoretical impasse was overcome by Prandtl, considered the father of modern fluid mechanics, who introduced in 1904 the concept of "boundary layer" [4]; a thin region near the solid surfaces where the effect of viscosity is dominant, and the velocity undergoes rapid change from zero at the wall to the value corresponding to the inviscid flow. Outside this region, the Euler equations apply.

On the other side, the pursuit to minimize the drag leads to more interest in natural laminar-flow wings, which are one of the most promising technologies for reducing fuel burn and emissions for commercial aviation [5-6].

Several resolutions to study laminar flow around airfoils have been conducted based on numerical simulations, experimental measurements or analytical approximations methods.

For example, the European SUPERTRAC (SUPERsonic TRAnsition Control) project, carried out numerical investigations within laminar flow technology applied to supersonic drag reduction specifically shape design optimization. The aim is to extend natural laminar flow over the wing surface and delay the start of laminar-turbulent transition [7-8].

Among the experimental measurements, a natural laminar supersonic flow wing, developed by the Japan Aerospace Exploration Agency, has been experimentally validated firstly in a supersonic wind tunnel and secondly by accomplishing flight tests [9].

In a theoretical facet, the asymptotic and perturbation methods provide powerful techniques for obtaining approximate solutions to complicated problems. For instance, the article [10] deals with modeling of supersonic flow around a dihedral airfoil for an inviscid compressible fluid, while [11] treats the asymptotic modeling of the aerodynamic coefficients of the NACA Airfoil in the case of laminar boundary layer flow for an incompressible and viscous fluid.

Using the mentioned methods [12-13], this paper is devoted to model the potential laminar flow of a real fluid over thin airfoils in a 2D curvilinear coordinate system to account for viscous effects; namely the development of the boundary layer and the interaction with the choc waves.

II. CASE STUDY SPECIFICATIONS

II.1 Assumptions

- **The fluid:**

The fluid considered here is compressible and viscous; beside it is non-heavy and Newtonian.

The fluid flow is considered laminar, two-dimensional, stationary, and isentropic.

The isentropy condition is justified by neglecting in energy equation the viscous dissipation term in the laminar boundary layer, the heat transfer term (Mach number <2) [14], and assuming weak shockwaves [15].

It's known that laminar boundary layer induces less friction than the turbulent one. For this reason, this model is more suitable to model laminar follows.

- **The wing profile:**

For this study, we choose thin NACA airfoils such as the relative curvature $\lambda = \frac{c}{R}$ (the ratio between the chord c and the radius of curvature R) is of order 1.

The study is carried out in the Body-fitted coordinate system.

The unit vectors are defined:

$\vec{e}$: Unit vector tangential to the wall of the profile.

$\vec{n}$: Unit vector normal to the profile wall.

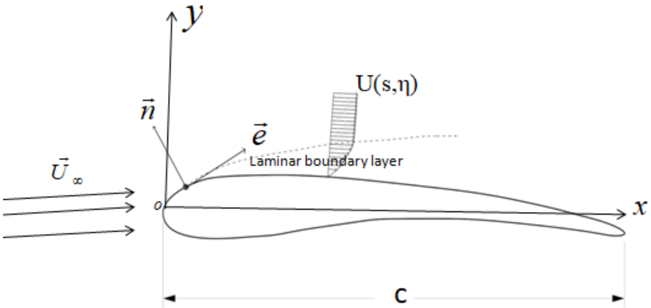

Figure 1: body fitted coordinate system

The velocity vector is: $\vec{u}(s,\eta) = U(s,\eta)\vec{e} + V(s,\eta)\vec{n}$

II.2 governing equations

The governing equations are the continuity, Navier-Stokes, and energy plus the thermodynamic state law of the fluid. For an isentropic steady flow of a perfect gas, we obtain the following system:

$$\begin{cases} div(\rho\vec{u}) = 0 \\ \rho\vec{u}\,\overrightarrow{\overrightarrow{grad}}(\vec{u}) = -\overrightarrow{grad}(P) + \mu\Delta\vec{u} \\ \rho C_P\vec{u}.\overrightarrow{grad}(T) = \vec{u}.\overrightarrow{grad}(P) \\ P = \rho RT \end{cases}$$

Bringing together these equations, the velocity field $\vec{u}(s,\eta)$ verifies:

$$\vec{u}.(\vec{u}.\overrightarrow{\overrightarrow{grad}}(\vec{u})) - c^2 div(\vec{u}) = \nu\vec{u}.\Delta(\vec{u}) \quad (1)$$

c: the speed of sound

υ: air dynamic viscosity

Expressing the mathematical operators: gradient, divergence, and Laplacian of the velocity vector field in curvilinear orthogonal coordinates system, eq.1 becomes:

$$\frac{\partial U}{\partial s}\left(\frac{U^2}{1-\eta/R} - \frac{C^2}{1-\eta/R}\right) + \frac{\partial V}{\partial \eta}(V^2 - C^2) + UV\left(\frac{1}{1-\eta/R}\frac{\partial V}{\partial s} + \frac{\partial U}{\partial \eta}\right) + \frac{V}{R(1-\eta/R)}C^2 =$$
$$\frac{\nu U}{1-\eta/R}\left[\frac{1}{1-\eta/R}\frac{\partial^2 U}{\partial s^2} - \frac{1}{R}\frac{\partial U}{\partial \eta} - \frac{1}{R(1-\eta/R)}\left(2\frac{\partial V}{\partial s} + \frac{U}{R}\right)\right] + \nu U\frac{\partial^2 U}{\partial \eta^2}$$
$$+\frac{\nu V}{1-\eta/R}\left[\frac{1}{1-\eta/R}\frac{\partial^2 V}{\partial s^2} - \frac{1}{R}\frac{\partial V}{\partial \eta} + \frac{1}{R(1-\eta/R)}\left(2\frac{\partial U}{\partial s} - \frac{V}{R}\right)\right] + \nu V\frac{\partial^2 V}{\partial \eta^2} \quad (2)$$

In case of irrotational flow, the velocity field derives from a potential: $\vec{u} = \overrightarrow{grad}(\phi(s,\eta))$

Hence, for a perfect fluid the equation 2 becomes the **stationary Steichen equation,** in 2D Cartesian coordinate system ($R \to \infty, \nu = 0$):

$$(1 - \frac{U^2}{C^2})\frac{\partial^2\phi}{\partial x^2} - 2\frac{UV}{C^2}\frac{\partial^2\phi}{\partial x\partial y} + (1 - \frac{V^2}{C^2})\frac{\partial^2\phi}{\partial y^2} = 0$$

II.3 Dimensionless equation

The dimensionless approach is the basis of asymptotic and perturbation methods using reference quantities:

U_0: the upstream velocity

c : the wing corde

Dimensionless quantities are expressed with a bar.

Hense, Eq.2 becomes:

$$\frac{1}{1-\lambda\bar{\eta}}\frac{\partial\bar{U}}{\partial\bar{s}}(\bar{U}^2 - \bar{C}^2) + \frac{\partial\bar{V}}{\partial\bar{\eta}}(\bar{V}^2 - \bar{C}^2) + \bar{U}\bar{V}\left(\frac{1}{1-\lambda\bar{\eta}}\frac{\partial\bar{V}}{\partial\bar{s}} + \frac{\partial\bar{U}}{\partial\bar{\eta}}\right) + \frac{\lambda\bar{V}}{(1-\lambda\bar{\eta})}\bar{C}^2$$
$$= \mathrm{Re}^{-1}\frac{\bar{U}}{(1-\lambda\bar{\eta})}\left[\frac{1}{1-\lambda\bar{\eta}}\frac{\partial^2\bar{U}}{\partial s^2} - \lambda\frac{\partial\bar{U}}{\partial\bar{\eta}} - \frac{\lambda}{(1-\lambda\bar{\eta})}\left(2\frac{\partial\bar{V}}{\partial\bar{s}} + \lambda\bar{U}\right)\right] + \mathrm{Re}^{-1}\bar{U}\frac{\partial^2\bar{U}}{\partial\bar{\eta}^2}$$
$$+\mathrm{Re}^{-1}\frac{\bar{V}}{(1-\lambda\bar{\eta})}\left[\frac{1}{1-\lambda\bar{\eta}}\frac{\partial^2\bar{V}}{\partial\bar{s}^2} - \lambda\frac{\partial\bar{V}}{\partial\bar{\eta}} + \frac{\lambda}{(1-\lambda\bar{\eta})}\left(2\frac{\partial\bar{U}}{\partial\bar{s}} - \lambda\bar{V}\right)\right] + \mathrm{Re}^{-1}\bar{V}\frac{\partial^2\bar{V}}{\partial\bar{\eta}^2} \quad (3)$$

Two parameters emerge:

- λ : the airfoil curvature,
- R_e : Reynolds number characterizes the flow regime.

II.4 Linearization process

Linearization constitutes a heuristic method to operate, under certain conditions, a simplification of the problem when $\varepsilon < 1$.

The associated linear problem is obtained once a basic solution to this problem is given. The solution is written as an asymptotic expansion:

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + ...$$

We consider airfoils with thickness: $\varepsilon < 1$ in near horizontal flow attack. The flow is attached to a large part of the airfoil (flow separation is not treated). Hence, the flow undergoes little disturbance.

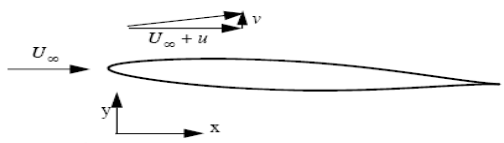

Figure 2: disturbed velocity field in the presence of a thin airfoil

a. Basic solution $\vec{\bar{U}}_\infty$

The linearization of the velocity field is done about a reference situation that is the upstream velocity U_∞.

At the upstream of the airfoil:

$$\begin{cases} s \to -\infty \\ \eta = \eta_\infty \end{cases} : \vec{\bar{U}}_\infty = \begin{bmatrix} \bar{U}_\infty \\ 0 \end{bmatrix}, \text{ such as: } \bar{U}_\infty = \frac{U_\infty}{U_0}$$

The non-perturbed velocity $\vec{\bar{U}}_\infty$ is a solution to the equation (3).We recognize two cases:

- **1st case:**

If we suppose the upstream airflow uniform and unidirectional: $\vec{\bar{U}}_\infty \begin{cases} \bar{U}_\infty = cte \\ \bar{V}_\infty = 0 \end{cases}$

Substitution in the dimensionless equation (3) leads to:

$$\mathrm{Re}^{-1}\bar{U}_\infty^2 = 0 \to \bar{U}_\infty = 0$$

This is a trivial solution.

- **2nd case:**

The base flow cannot have uniform shear but varies with altitude.

As we are dealing with laminar flow over airfoils in cruising flight, we consider the standard atmosphere.

Therefore, we do not introduce instabilities such as: turbulence, wind, gust... [16].

The basic flow $\vec{\bar{U}}_\infty$ is chosen to be particularly simple, so that $\bar{U}_\infty$ depends only on $\bar{\eta}$:

we get this Cauchy–Euler equation:

$$(1-\bar{\eta})^2 \frac{\partial^2 \bar{U}_\infty}{\partial \bar{\eta}^2} - (1-\bar{\eta}) \frac{\partial \bar{U}_\infty}{\partial \bar{\eta}} - \bar{U}_\infty = 0$$

we assume a linear relationship between $\overline{\eta_\infty}$ and $\bar{\eta}$, since the flow is little disturbed by the thin profile:

$$\overline{\eta_\infty} \cong \alpha_0 \bar{\eta} + o(\bar{\eta}^2) \text{ with } \bar{\eta}_\infty = \frac{\eta_\infty}{\eta_\infty^0}$$

$\bar{\eta}$ is the normal coordinate of the body fitted curvilinear coordinate system.

η_∞ is the altitude with respect to the standard atmosphere.

$\eta_\infty{}^0$ is the reference altitude (defined afterward).

Hence: $\bar{U}_\infty = \bar{U}_\infty(\bar{\eta}_\infty)$

verifies the dimensionless equation:

$$(\alpha_0 - \bar{\eta}_\infty)^2 \frac{\partial^2 \bar{U}_\infty}{\partial \bar{\eta}_\infty^2} - (\alpha_0 - \bar{\eta}_\infty) \frac{\partial \bar{U}_\infty}{\partial \bar{\eta}_\infty} - \bar{U}_\infty = 0$$

The solution is:

$$\bar{U}_\infty(\bar{\eta}_\infty) = C_1(\bar{\eta}_\infty - \alpha_0) + \frac{C_2}{\bar{\eta}_\infty - \alpha_0}$$

C_1 and C_2 are two constants.

Since the velocity field associated with $\vec{\bar{U}}_\infty$ is irrotational; the Rotational pseudo-vector is equal to zero:

Rotational tensor:

$$\vec{\vec{Rot}}(\bar{u}_\infty) = \begin{bmatrix} 0 & \frac{\partial U}{\partial \eta} - \frac{1}{1-\frac{\eta}{R}}\frac{U}{R} & 0 \\ \frac{\partial U}{\partial \eta} - \frac{1}{1-\frac{\eta}{R}}\frac{U}{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The pseudo Rotational Vector is:

$$\overrightarrow{Rot}\ \bar{u} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial U}{\partial \eta} - \frac{1}{1-\frac{\eta}{R}}\frac{U}{R} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In dimensionless form:

$$\alpha_0 \frac{\partial \bar{U}}{\partial \bar{\eta}_\infty} - \frac{\bar{U}}{1-\frac{\bar{\eta}_\infty}{\alpha_0}} = 0 \quad (\bar{R} = \lambda^{-1} = 1)$$

Solution of this equation corresponds to the rotational component:

$$\overline{U}_\infty^{rotationnal}(\bar{\eta}_\infty) = \frac{C_2}{\alpha_0 - \bar{\eta}_\infty}$$

Lastly: $\bar{U}_\infty(\bar{\eta}_\infty) = C_1(\bar{\eta}_\infty - \alpha_0)$ (*)

In the standard atmosphere, we have:

$$\begin{cases} T_\infty(\eta_\infty) = 288{,}15 - 6{,}5\eta_\infty; \quad 0 \le \eta_\infty \le 11km \\ T_\infty(\eta_\infty) = 216{,}65°K; \quad 12km \le \eta_\infty \le 20km \end{cases}$$

Above 12 km, the speed of sound: C is constant. This is defined as the reference velocity:

$$U_0 = C^*_\infty(12km \le \eta_\infty \le 20km) = 295{,}069 m/s$$

The reference altitude is: $\eta^0{}_\infty = 20km$.

Then, the following table is obtained for the dimensionless speed of sound.

Table1: adimensional speed of sound according to the altitude

$\overline{\eta_\infty}$	0	0,05	0,1	0,15	0,2	0,25	0,3
$\overline{c_\infty}(\overline{\eta_\infty})$	1,15327	1,14019	1,12696	1,11358	1,10004	1,08634	1,07247
$\overline{\eta_\infty}$	0,35	0,4	0,45	0,5	0,55	0,6	0,65
$\overline{c_\infty}(\overline{\eta_\infty})$	1,05842	1,04418	1,02975	1,01512	1,00028	1	1
$\overline{\eta_\infty}$	0,7	0,75	0,8	0,85	0,9	0,95	1
$\overline{c_\infty}(\overline{\eta_\infty})$	1	1	1	1	1	1	1

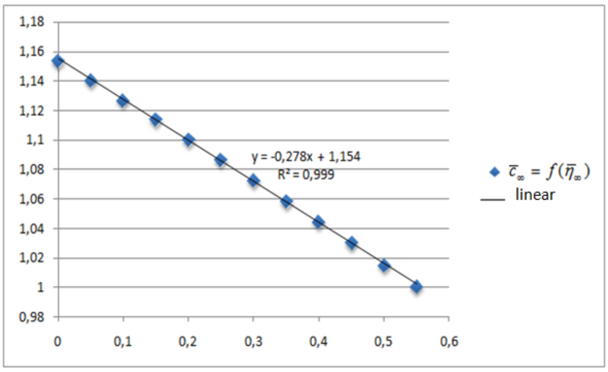

Figure 3: linear regression curve of the dimensionless speed of sound

A linear regression model of the dimensionless speed of sound as a function of altitude is found, with a correlation of 99%, for the speed of sound up to 11km, confirms the correctness of the approximation (*).

Thus**:**

$$\begin{cases} \bar{U}_\infty(\bar{\eta}) = M_\infty(-0,278\bar{\eta}_\infty + 1,154), 0 \le \bar{\eta}_\infty \le 0.55 \ (**) \\ \bar{U}_\infty(\bar{\eta}) = M_\infty ; 0.6 \le \bar{\eta}_\infty \le 1 \end{cases}$$

From (*) and (**), we get the expression of the upstream velocity, using η coordinate, up to 11km:

$$\begin{cases} \bar{U}_\infty(\bar{\eta}) = -1,154 M_\infty(\bar{\eta} - 1) \ ; \ 0 \le \bar{\eta} \le 0.1325 \\ \bar{U}_\infty(\bar{\eta}) = M_\infty \quad ; \quad 0.1445 \le \bar{\eta} \le 0.241 \end{cases} \quad (4)$$

$$\alpha_0 = 4.151$$

b. Linearization with regard to $\vec{\bar{U}}_\infty$

The thikness is small, so the aerofoil will produce small perturbations in the surrounding flow.

We look for the corresponding solution in the form of the straightforward asymptotic expansions.

In dimensionless form, velocity components of $\vec{u}(U,V)$ are:

$$\bar{U} = \bar{U}_\infty + \frac{\varepsilon}{1-\lambda\bar{\eta}}\bar{\varphi}_{\bar{s}}$$

$$\bar{V} = \varepsilon\bar{\varphi}_{\bar{\eta}}$$

In eq.3, to have only the components of the velocity the speed of sound is replaced by the local air velocity using the Bernoulli relation:

$$\frac{u^2}{2} + \frac{c^2}{\gamma - 1} = cte$$

Then Bernoulli's theorem applied along a streamline in dimensionless form leads to:

$$\bar{C}^2 = \frac{C^2}{U_0{}^2} = M_\infty{}^{-2}\bar{U}_\infty{}^2 - \frac{\gamma-1}{2}(\bar{U}^2 + \bar{V}^2 - \bar{U}_\infty{}^2) =$$

$$M_\infty{}^{-2}\bar{U}_\infty{}^2 - \frac{\gamma-1}{2}\left[2\bar{U}_\infty\frac{\varepsilon\bar{\varphi}_{\bar{s}}}{1-\lambda\bar{\eta}} + \left(\frac{\varepsilon\bar{\varphi}_{\bar{s}}}{1-\lambda\bar{\eta}}\right)^2 + (\varepsilon\bar{\varphi}_{\bar{\eta}})^2\right]$$

c. Asymptotic expansion model of potential laminar flow

Substituting the above dimensionless velocities: U, V and C into (eq.3), an asymptotic expansion up to order 3 of the linearized perturbed velocity potential equation is obtained:

$$\varepsilon^2\left\{\left[\frac{(\gamma+1)\bar{U}_\infty\bar{\varphi}_{\bar{s}\bar{s}}\bar{\varphi}_{\bar{s}}}{1-\lambda\bar{\eta}} + (1-\lambda\bar{\eta})(\gamma-1)\bar{U}_\infty\bar{\varphi}_{\bar{s}}\bar{\varphi}_{\bar{\eta}\bar{\eta}} + (1-\lambda\bar{\eta})\bar{\varphi}_{\bar{s}}\bar{\varphi}_{\bar{\eta}}\frac{\partial\bar{U}_\infty}{\partial\bar{\eta}} + (1-\lambda\bar{\eta})\bar{U}_\infty\bar{\varphi}_{\bar{\eta}}\left(\bar{\varphi}_{\bar{s}\bar{\eta}} + \bar{\varphi}_{\bar{\eta}\bar{s}} + \frac{\lambda\bar{\varphi}_{\bar{s}}}{1-\lambda\bar{\eta}}\right) - \lambda(\gamma-1)\bar{U}_\infty\bar{\varphi}_{\bar{\eta}}\bar{\varphi}_{\bar{s}}\right]\right.$$

$$\left. -\mathrm{Re}^{-1}\left[\bar{\varphi}_{\bar{s}}\left[\bar{\varphi}_{\bar{\eta}\bar{\eta}\bar{s}} + \frac{\bar{\varphi}_{\bar{s}\bar{s}\bar{s}}}{(1-\lambda\bar{\eta})^2} - \lambda\left(\frac{\bar{\varphi}_{\bar{\eta}\bar{s}}}{1-\lambda\bar{\eta}}\right)\right] + \bar{\varphi}_{\bar{\eta}}\left[\bar{\varphi}_{\bar{s}\bar{s}\bar{\eta}} - \lambda(1-\lambda\bar{\eta})\bar{\varphi}_{\bar{\eta}\bar{\eta}} + 2\lambda\frac{\bar{\varphi}_{\bar{s}\bar{s}}}{1-\lambda\bar{\eta}} - \lambda^2\bar{\varphi}_{\bar{\eta}}\right] + (1-\lambda\bar{\eta})^2\bar{\varphi}_{\bar{\eta}}\bar{\varphi}_{\bar{\eta}\bar{\eta}\bar{\eta}}\right]\right\}$$

$$+\varepsilon^1\left\{\left[\bar{\varphi}_{\bar{s}\bar{s}}\bar{U}_\infty{}^2(1-M_\infty{}^{-2}) - (1-\lambda\bar{\eta})^2\bar{\varphi}_{\bar{\eta}\bar{\eta}}\bar{U}_\infty{}^2 M_\infty{}^{-2} + (1-\lambda\bar{\eta})^2\frac{\partial\bar{U}_\infty}{\partial\bar{\eta}}\bar{U}_\infty\bar{\varphi}_{\bar{\eta}} + \lambda(1-\lambda\bar{\eta})\bar{U}_\infty{}^2 M_\infty{}^{-2}\bar{\varphi}_{\bar{\eta}}\right]\right.$$

$$\left. -\mathrm{Re}^{-1}\bar{U}_\infty\left(\frac{\bar{\varphi}_{\bar{s}\bar{s}\bar{s}}}{(1-\lambda\bar{\eta})} + (1-\lambda\bar{\eta})\bar{\varphi}_{\bar{\eta}\bar{\eta}\bar{s}} - \lambda\bar{\varphi}_{\bar{\eta}\bar{s}}\right)\right\}$$

$$-\varepsilon^0\,\mathrm{Re}^{-1}\bar{U}_\infty\left[(1-\lambda\bar{\eta})^2\frac{\partial^2\bar{U}_\infty}{\partial\bar{\eta}^2} - \lambda(1-\lambda\bar{\eta})\frac{\partial\bar{U}_\infty}{\partial\bar{\eta}} - \lambda^2\bar{U}_\infty\right] + o(\varepsilon^3) = 0 \quad (5)$$

This model is written in the form:

$$L(\bar{\varphi}) = L_0(\bar{\varphi}) + \varepsilon L_1(\bar{\varphi}) + \varepsilon^2 L_2(\bar{\varphi}) + o(\varepsilon^3) = 0 \ ; \ \ \bar{\varphi} = \bar{\varphi}(\bar{s}, \bar{\eta}, \varepsilon, \lambda, M_\infty, \mathrm{Re}^{-1})$$

This equation models the potential of small disturbances of steady subsonic to supersonic airflow with Mach numbers up to 2.

It depends on the spacial variables (s,η), Reyolds number, Mach number, and body shape (λ, ε).

For airfoils with λ=O(1) using the irrotationality condition for the upstream velocity:

$$(1-\bar{\eta})\frac{\partial\bar{U}_\infty}{\partial\bar{\eta}} = \bar{U}_\infty$$

eq.5 is simplified:

At order 1:

$$(M_\infty{}^2 - 1)\bar{\varphi}_{\bar{s}\bar{s}} - (1-\bar{\eta})^2\bar{\varphi}_{\bar{\eta}\bar{\eta}} + (M_\infty{}^2 + 1)(1-\bar{\eta})\bar{\varphi}_{\bar{\eta}}$$

$$+\mathrm{Re}^{-1}\frac{M_\infty{}^2}{\bar{U}_\infty}\left[\frac{\bar{\varphi}_{\bar{s}\bar{s}\bar{s}}}{(1-\bar{\eta})} + (1-\bar{\eta})\bar{\varphi}_{\bar{\eta}\bar{\eta}\bar{s}} - \bar{\varphi}_{\bar{\eta}\bar{s}}\right] = 0$$

At order 2:

$$M_\infty{}^2\left[\frac{\gamma+1}{1-\bar{\eta}}\frac{\bar{\varphi}_{\bar{s}\bar{s}}\bar{\varphi}_{\bar{s}}}{\bar{U}_\infty} + (1-\bar{\eta})(\gamma-1)\frac{\bar{\varphi}_{\bar{\eta}\bar{\eta}}\bar{\varphi}_{\bar{s}}}{\bar{U}_\infty} + 2(1-\bar{\eta})\frac{\bar{\varphi}_{\bar{s}\bar{\eta}}\bar{\varphi}_{\bar{\eta}}}{\bar{U}_\infty} + (3-\gamma)\frac{\bar{\varphi}_{\bar{s}}\bar{\varphi}_{\bar{\eta}}}{\bar{U}_\infty}\right] -$$

$$\mathrm{Re}^{-1}\frac{M_\infty{}^2}{\bar{U}_\infty{}^2}\left[\begin{array}{l} \bar{\varphi}_{\bar{s}}\bar{\varphi}_{\bar{\eta}\bar{\eta}\bar{s}} + \dfrac{\bar{\varphi}_{\bar{s}}\bar{\varphi}_{\bar{s}\bar{s}\bar{s}}}{(1-\bar{\eta})^2} - \left(\dfrac{\bar{\varphi}_{\bar{s}}\bar{\varphi}_{\bar{\eta}\bar{s}}}{1-\bar{\eta}}\right) + \bar{\varphi}_{\bar{\eta}}\bar{\varphi}_{\bar{s}\bar{s}\bar{\eta}} \\ -(1-\bar{\eta})\bar{\varphi}_{\bar{\eta}}\bar{\varphi}_{\bar{\eta}\bar{\eta}} + 2\dfrac{\bar{\varphi}_{\bar{\eta}}\bar{\varphi}_{\bar{s}\bar{s}}}{1-\bar{\eta}} - \bar{\varphi}_{\bar{\eta}}{}^2 + (1-\bar{\eta})^2\bar{\varphi}_{\bar{\eta}}\bar{\varphi}_{\bar{\eta}\bar{\eta}\bar{\eta}} \end{array}\right] = 0$$

According to small perturbation theory ($\frac{\bar{\varphi}_{\bar{s}}}{\bar{U}_\infty} \ll 1$ & $\frac{\bar{\varphi}_{\bar{\eta}}}{\bar{U}_\infty} \ll 1$); terms of order 2 are negligible compared to order 1for subsonic (M<0.8) and supersonic (1.2<M<5) flows. Furthermore, for transonic flow (0.8<M<1.2), viscous terms of order 2 are negligible compared to those of order 1, but the second order compressibility term: $M_{\infty}^{2}\frac{\gamma+1}{1-\bar{\eta}}\frac{\bar{\varphi}_{\bar{s}\bar{s}}\bar{\varphi}_{\bar{s}}}{\bar{U}_{\infty}}$ can't be neglected.

Consequently, the following singular perturbed models are deduced:

- Subsonic and supersonic range:

$$(M_{\infty}^{2}-1)(1-\bar{\eta})\bar{\varphi}_{\bar{s}\bar{s}}-(1-\bar{\eta})^{3}\bar{\varphi}_{\bar{\eta}\bar{\eta}}+(1-\bar{\eta})^{2}(M_{\infty}^{2}+1)\bar{\varphi}_{\bar{\eta}}$$
$$-0.866M_{\infty}\,\mathrm{Re}^{-1}\left[\frac{\bar{\varphi}_{\bar{s}\bar{s}\bar{s}}}{(1-\bar{\eta})}+(1-\bar{\eta})\bar{\varphi}_{\bar{\eta}\bar{\eta}\bar{s}}-\bar{\varphi}_{\bar{\eta}\bar{s}}\right]=0 \qquad (6)$$

- Transonic range:

$$[(M_{\infty}^{2}-1)+\varepsilon M_{\infty}^{2}\frac{(\gamma+1)\bar{\varphi}_{\bar{s}}}{(1-\bar{\eta})\bar{U}_{\infty}}]\bar{\varphi}_{\bar{s}\bar{s}}-(1-\bar{\eta})^{2}\bar{\varphi}_{\bar{\eta}\bar{\eta}}+(1-\bar{\eta})(M_{\infty}^{2}+1)\bar{\varphi}_{\bar{\eta}}$$
$$-0.866\frac{M_{\infty}\,\mathrm{Re}^{-1}}{(1-\bar{\eta})}\left[\frac{\bar{\varphi}_{\bar{s}\bar{s}\bar{s}}}{(1-\bar{\eta})}+(1-\bar{\eta})\bar{\varphi}_{\bar{\eta}\bar{\eta}\bar{s}}-\bar{\varphi}_{\bar{\eta}\bar{s}}\right]=0 \qquad (7)$$

These singular perturbation models are valid for curved bodies, unlike the classical models below for a flat plate. Furthermore they introduce real fluid phenomena; that is the development of the boundary layer, the formation of choc waves, and their interaction.

III. COMPARISON WITH EXISTING MODELS

a. Subosnic and sueprsonic inviscid flow

The new general formulation of the potential steady flow equation in subsonic and supersonic range for an inviscid fluid, expressed in the orthogonal curvilinear coordinate system, taking into account the body curvature λ (eq.5) is:

$$(M_{\infty}^{2}-1)\bar{\varphi}_{\bar{s}\bar{s}}-(1-\lambda\bar{\eta})^{2}\bar{\varphi}_{\bar{\eta}\bar{\eta}}+(1-\lambda\bar{\eta})(M_{\infty}^{2}+\lambda)\bar{\varphi}_{\bar{\eta}}=0$$

Meanwhile, the famous formula expressed in Cartesian coordinate system for an inviscid flow over a flat plate is:

$$(M_{\infty}^{2}-1)\bar{\varphi}_{xx}-\bar{\varphi}_{yy}=0$$

If we assume further that $\bar{\varphi}_{\bar{\eta}}\ll 1$, according to the small perturbation assumption, *s*etting λ=0 (for a flat plate), and for low Mach numbers (M<2), this term $(1-\lambda\bar{\eta})(M_{\infty}^{2}+\lambda)\bar{\varphi}_{\bar{\eta}}$ may be neglected.

The plate model is a particular case of the developed model.

b. Transonic inviscid flow

The new general model formula (eq.5), without viscous terms, is:

$$[1-M_{\infty}^{2}-\varepsilon\frac{(\gamma+1)M_{\infty}^{2}}{(1-\lambda\bar{\eta})\bar{U}_{\infty}}\bar{\varphi}_{\bar{s}}]\bar{\varphi}_{\bar{s}\bar{s}}+(1-\lambda\bar{\eta})^{2}\bar{\varphi}_{\bar{\eta}\bar{\eta}}+(1-\lambda\bar{\eta})(M_{\infty}^{2}+\lambda)\bar{\varphi}_{\bar{\eta}}=0$$

The well-known transonic small disturbance(TSD) equation in cartesian coordonate system is:

$$(1-M_{\infty}^{2}-\frac{(\gamma+1)M_{\infty}^{2}}{\bar{U}_{\infty}}\bar{\varphi}_{x})\bar{\varphi}_{xx}+\bar{\varphi}_{yy}=0$$

Again, there is a similarity between the two models.

The airfoil thickness ε appears in the new developed equation as we seek an approached asymptotic model.

CONCLUSION

By applying asymptotic and perturbation methods, we derive an approached analytic model for a steady, two-dimensional, irrotational, and laminar flow of a compressible and viscous fluid for curved bodies.

This singular perturbation model, constructed with respect to the standard atmosphere, is more suitable than existing ones for subsonic to supersonic range flow computations, and for the more appropriate modeling and analysis of viscous effects.

This model may be used to study supersonic laminar flow wings.

REFERENCES

[1] Jean le Rond d'Alembert, Essai d'une nouvelle théorie de la résistance des fluides(Essay of A New Theory of Fluid Resistance), Paris, 1752 (in French).
[2] Genesis of d'Alembert's paradox and analytical elaboration of the drag problem
[3] A. Saint-Venant, Résistance des fluides: considérations historiques, physiques et pratiques relatives au problème de l'action dynamique mutuelle d'un fluide a d'un solide, dans l'état de permanence supposé acquis par leurs mouvements mouvements (Fluid resistance. Historical and practical considerations relative to the problem of the mutual dynamic action of a fluid and a solid, especially under steady conditions supposed obtained by its movement), Académie des Science, Mémoires 44, 1-280 (in French).
[4] L. Prandtl. On motion of fluids with very little viscosity. Third International Congress of Mathematics, Heidelberg, http://naca.larc.nasa.gov/digidoc/report/tm/52/NACA-TM-452.PDF,1904.]
[5] Géza Schrauf. Status and perspectives of laminar flow. Aeronautical Journal -New Series- 109(1102):639-644 DOI:10.1017/S000192400000097X. December 2005
[6] Nils Beck, Tim Landa, Arne Seitz, Loek Boermans, Yaolong Liu and Rolf Radespiel. Drag Reduction by Laminar Flow Control. Energies 2018, 11, 252; doi:10.3390/en11010252
[7] Emiliano Iuliano, Itham Salah El Din, Raffaele Salvatore Donelli, D. Quagliarella, Natural Laminar Flow Design of a Supersonic Transport Jet Wing Body. Conference: 47th AIAA Aerospace Sciences Meeting including The New Horizons Forum and Aerospace Exposition (January 2009). DOI:10.2514/6.2009-1279
[8] D. Arnal, C.G. Unckel, J. Krier, J.M. Sousa, S. Hein, Supersonic laminar flow control studies In the SUPERTRAC project. 25th International congress of the Aeronautical Sciences. ICAS 2006 (September 2006).
[9] Olivier Vermeersch, Kenji Yoshida, Yoshine Ueda, Daniel Arnal. Natural laminar flow wing for supersonic conditions: Wind tunnel experiments, flight test and stability computations. Progress in Aerospace Sciences 79(2015)64–91
[10] A. Naamane, M. Hasnaoui, Supersonic Flow around a Dihedral Airfoil: Modeling and Experimentation Investigation, World Academy of Science, Engineering and Technology International Journal of Aerospace and Mechanical Engineering Vol: 13, No: 6, (2019)
[11] M.Hasnaoui, A.Naamane, H.Akhmari, Asymptotic modeling the Aerodynamic coefficients of the NACA Airfoil. Modeling, IIETA Journals, Measurement and Control B Vol. 88, No. 2-4, Page: 58-66 (2019)
[12] Radyadour KH. Zeytounian, Asymptotic Modeling of Fluid Flow Phenomena. (KLUWER ACADEMIC PUBLISHERS, 2002).

[13] Jean Cousteix & Jacques Mauss. Analyse asymptotique et couche limite. (Springer, 2006)
[14] Patrick H. Oosthuizen, William E. Carscallen, Introduction to Compressible Fluid Flow, second edition, p 310 (2013)
[15] José Pontes, Norberto Mangiavacchi, Gustavo Rabello dos Anjos, An Introduction to Compressible Flows with Applications Quasi One Dimensional Approximation and General Formulation for Subsonic, Transonic, and Supersonic Flows p. 43 (Springer International Publishing, 2019).
[16] Andreas Daniel Reeh. Natural Laminar Flow Airfoil Behavior in Cruise Flight through Atmospheric Turbulence, Ph.D. dissertation, Mecha. Eng. Dept., Darmstadt Univ., (2014